Adaptive Modelling of Complex Data: Kernels

Part 2: Convex optimization, support vector machines

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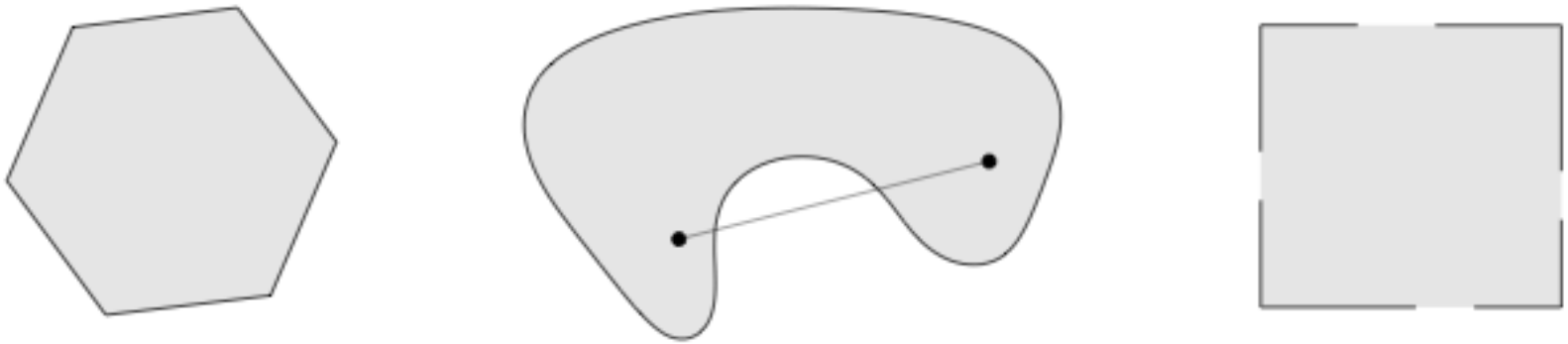
Overview

- Review of convex optimization
- Support vector classification, the C-SV machine
- The representer theorem

Short overview of convex optimization

Convex set



(Figure from Boyd and Vandenberghe)

Leftmost set is convex, remaining two are not.

Every point in the set can be seen from any other point in the set, along a straight line that never leaves the set.

Definition

C is convex if for all $x_1, x_2 \in C$ and any $0 \leq \theta \leq 1$ we have $\theta x_1 + (1-\theta)x_2 \in C$, i.e. every point on the line between x_1 and x_2 lies in C.

Convex function: no local optima

(Figure from Boyd and Vandenberghe)

Definition (Convex function)

A function f is **convex** if its domain $\mathrm{dom} f$ is a convex set and if $\forall x, y \in \mathrm{dom} f$, and any $0 \leq \theta \leq 1$,

$$f\left(\theta x + (1-\theta)y\right) \leq \theta f(x) + (1-\theta) f(y).$$

The function is **strictly convex** if the inequality is strict for $x \neq y$.

Optimization and the Lagrangian

Optimization problem on $x \in \mathbb{R}^n$ / primal,

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & i = 1, \ldots, m \\ & h_j(x) = 0 & j = 1, \ldots p. \end{array} \qquad (1)$$

- domain $\mathcal{D} := \bigcap_{i=0}^{m} \mathrm{dom} f_i \cap \bigcap_{j=1}^{p} \mathrm{dom} h_j$ (nonempty).
- p^* the optimal value of (1)

Idealy we would want an unconstrained problem

$$\text{minimize } f_0(x) + \sum_{i=1}^{m} I_- \left(f_i(x)\right) + \sum_{j=1}^{p} I_0 \left(h_j(x)\right),$$

where $I_-(u) = \begin{cases} 0, & u \leq 0 \\ \infty, & u > 0 \end{cases}$ and $I_0(u) = \begin{cases} 0, & u = 0 \\ \infty, & u \neq 0 \end{cases}$.

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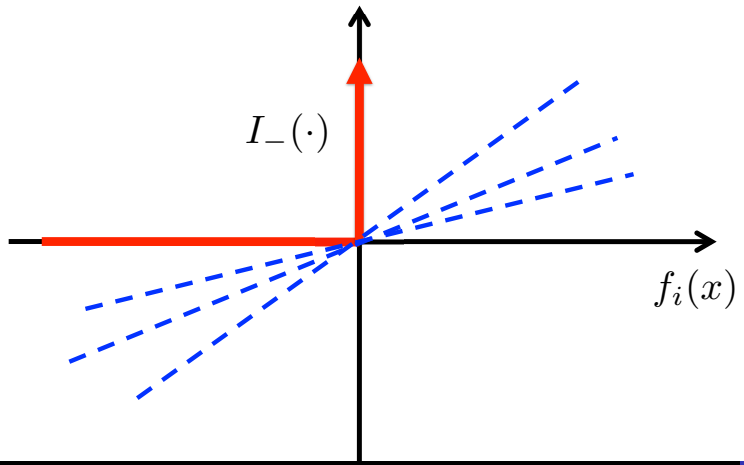
Lower bound interpretation of Lagrangian

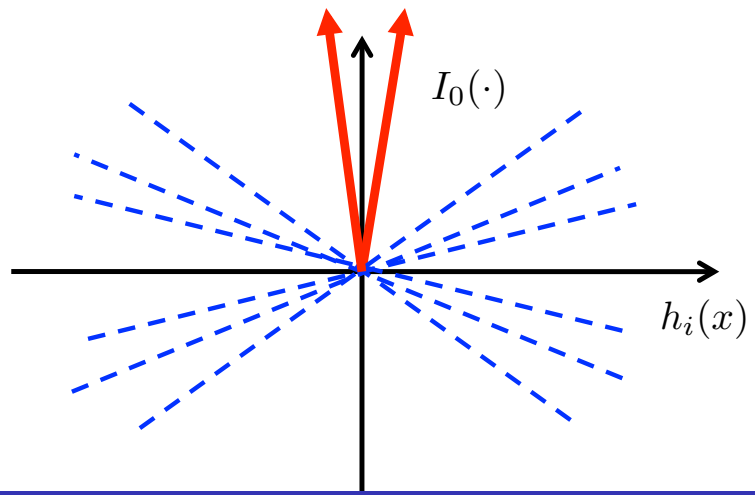
The **Lagrangian** $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ is an (easier to optimize) lower bound on the original problem:

$$L(x, \lambda, \nu) := f_0(x) + \sum_{i=1}^{m} \underbrace{\lambda_i f_i(x)}_{\leq I_-(f_i(x))} + \sum_{j=1}^{p} \underbrace{\nu_j h_j(x)}_{\leq I_0\left(h_j(x)\right)},$$

and has domain $\mathrm{dom} L := \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p$. The vectors λ and ν are called **Lagrange multipliers** or **dual variables**.

To ensure a lower bound, we require $\lambda \succeq 0$.





Lagrange dual: lower bound on optimum p^*

The **Lagrange dual function:** minimize Lagrangian
When $\lambda \succeq 0$ and $f_i(x) \leq 0$, Lagrange dual function is

$$g(\lambda, \nu) := \inf_{x \in \mathcal{D}} L(x, \lambda, \nu). \tag{2}$$

A **dual feasible** pair (λ, ν) is a pair for which $\lambda \succeq 0$ and $(\lambda, \nu) \in \mathrm{dom}(g)$.
We will show: (next slide) for any $\lambda \succeq 0$ and ν,

$$g(\lambda, \nu) \leq f_0(x)$$

wherever

$$\begin{aligned} f_i(x) &\leq 0 \\ h_j(x) &= 0 \end{aligned}$$

(including at $f_0(x^*) = p^*$).

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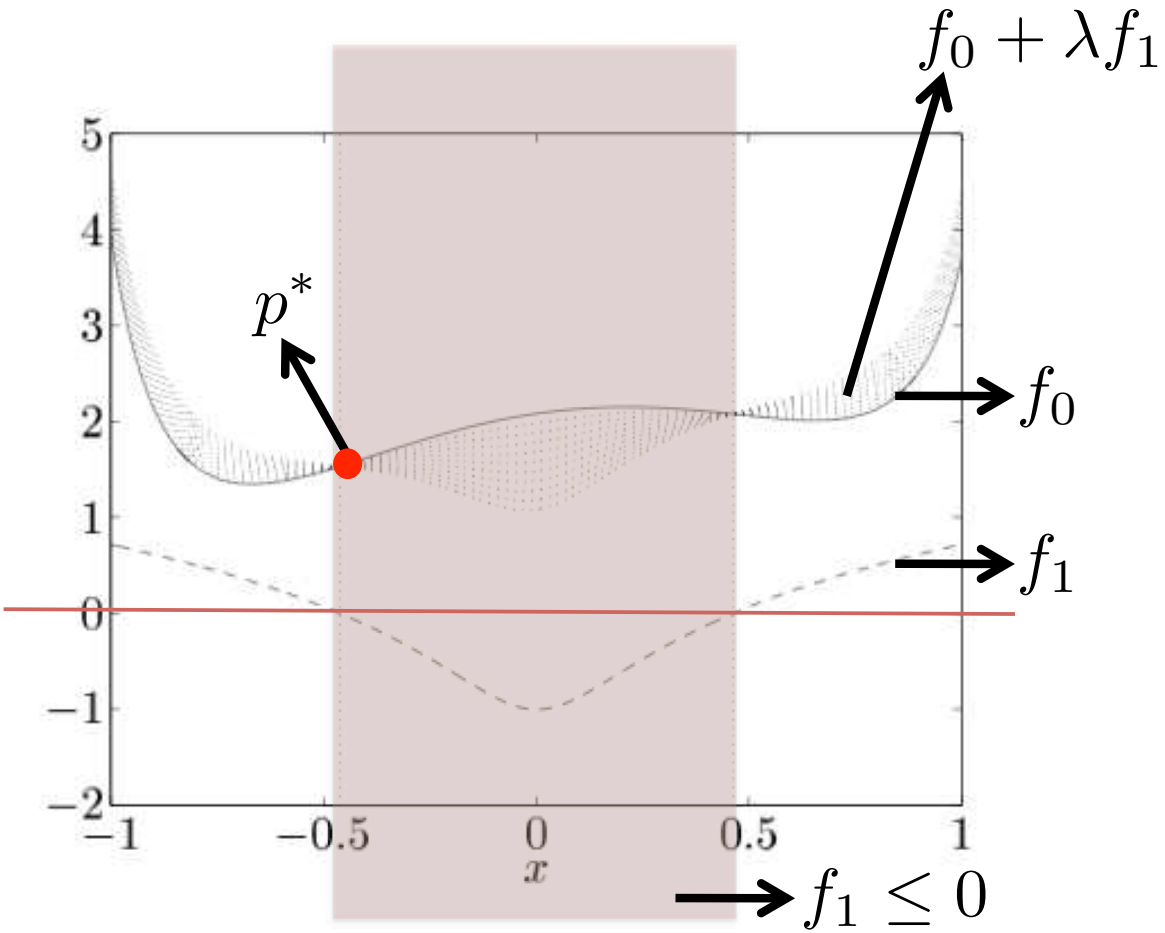
Simplest example: minimize over x the function

$L(x, \lambda) = f_0(x) + \lambda f_1(x)$

(Figure from Boyd and Vandenberghe)



Reminders:

- f_0 is function to be minimized.
- $f_1 \leq 0$ is inequality constraint
- $\lambda \geq 0$ is Lagrange multiplier
- p^* is minimum f_0 *in constraint set*

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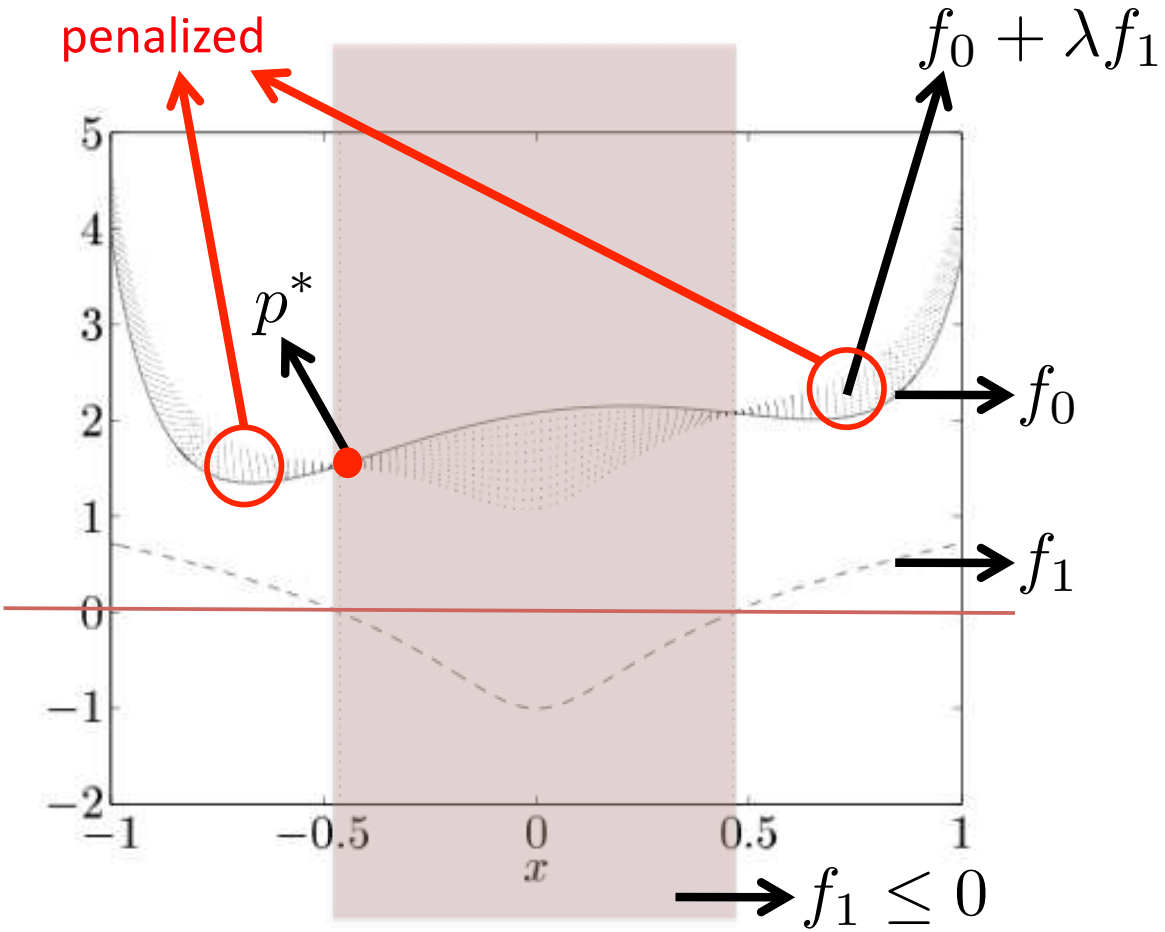

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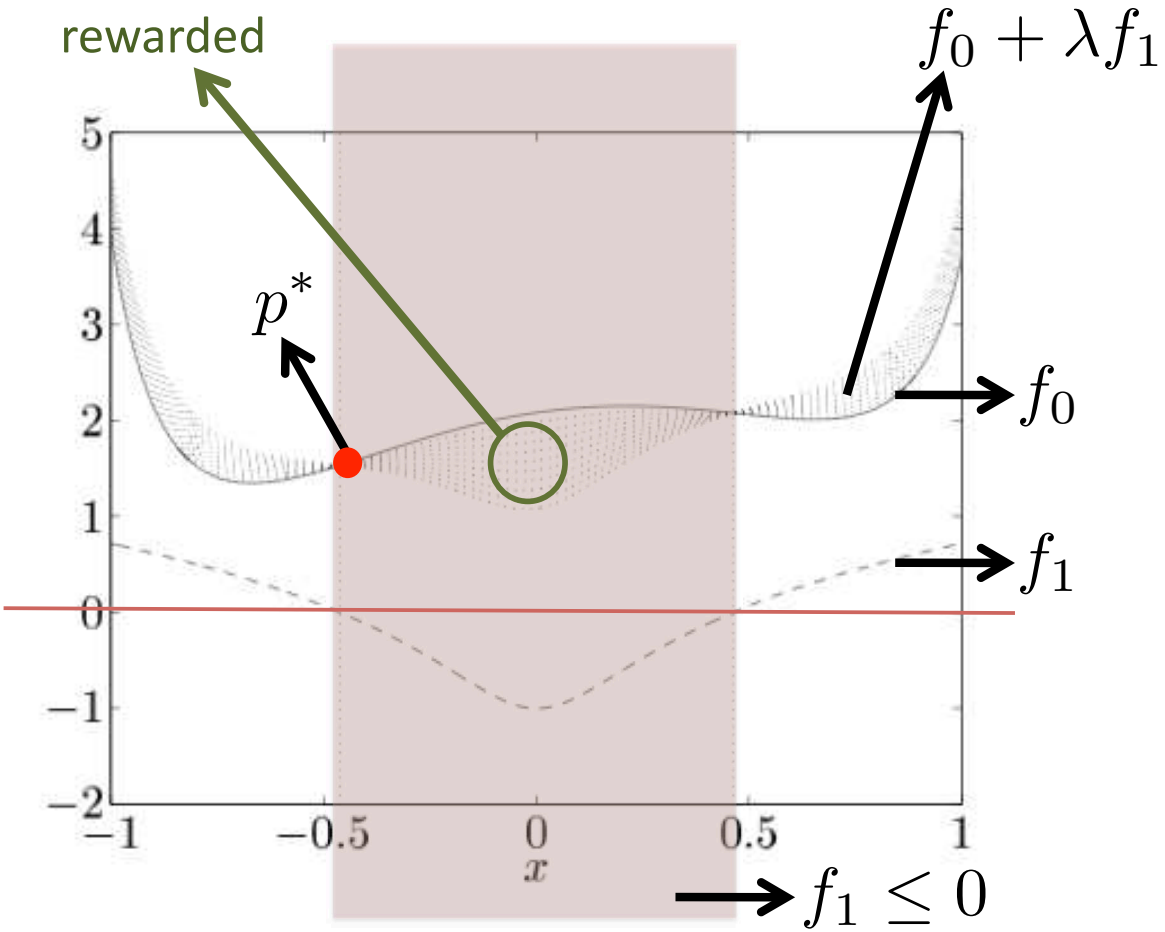

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Lagrange dual is lower bound on p^* (proof)

We now give a formal proof that **Lagrange dual function** $g(\lambda, \nu)$ lower bounds p^*.

Proof: Assume $\tilde{x}$ is feasible, i.e. $f_i(\tilde{x}) \leq 0$, $h_i(\tilde{x}) = 0$, $\tilde{x} \in \mathcal{D}$, $\lambda \succeq 0$. Then

$$\sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \leq 0$$

Thus

$$\begin{aligned} g(\lambda, \nu) &:= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right) \\ &\leq f_0(\tilde{x}) + \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \\ &\leq f_0(\tilde{x}). \end{aligned}$$

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Best lower bound: maximize the dual

Best (i.e. biggest) lower bound $g(\lambda, \nu)$ on the optimal solution p^* of original problem: **Lagrange dual problem**

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0. \end{array} \tag{3}$$

Dual feasible: (λ, ν) with $\lambda \succeq 0$ and $g(\lambda, \nu) > -\infty$.
Dual optimal: solutions (λ^*, ν^*) to the dual problem, d^* is optimal value (**dual always easy to maximize**: next slide).
Weak duality always holds:

$$d^* \leq p^*.$$

...but what is the point of finding a **biggest lower bound** on a **minimization problem**?

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Strong duality: (does **not** always hold, conditions given later):

$$d^* = p^*.$$

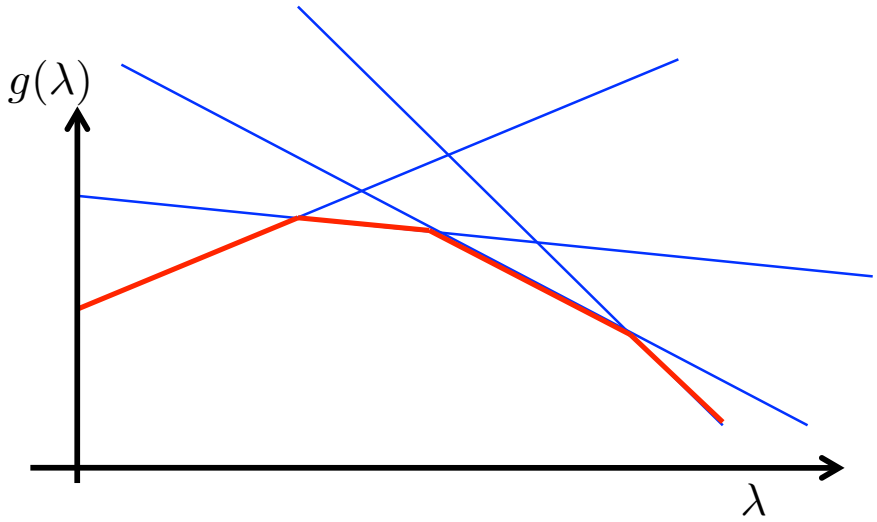
If S.D. holds: solve the **easy (concave) dual problem** to find p^*.

Maximizing the dual is always easy

The **Lagrange dual function:** minimize Lagrangian (lower bound)

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu).$$

Dual function is a pointwise infimum of affine functions of (λ, ν), hence **concave** in (λ, ν) with convex constraint set $\lambda \succeq 0$.



Example:

One inequality constraint,

$$L(x, \lambda) = f_0(x) + \lambda f_1(x),$$

and assume there are only four possible values for x. Each line represents a different x.

How do we know if strong duality holds?

Conditions under which strong duality holds are called **constraint qualifications** (they are sufficient, but not necessary)

(Probably) best known sufficient condition: Strong duality holds if

- Primal problem is **convex**, i.e. of the form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \qquad\qquad i = 1, \ldots, n \\ & Ax = b \end{array}$$

for convex $f_0, \ldots, f_m$, *and*

Slater's condition: there exists a strictly feasible point $\tilde{x}$, such that $f_i(\tilde{x}) < 0$, $i = 1, \ldots, n$ (reduces to the existence of any feasible point when inequality constraints are affine, i.e., $Cx \preceq d$).

A consequence of strong duality...

Assume primal is equal to the dual. What are the consequences?

- x^* solution of original problem (minimum of f_0 under constraints),
- (λ^*, ν^*) solutions to dual

$$
\begin{aligned}
f_0(x^*) &\underset{\text{(assumed)}}{=} g(\lambda^*, \nu^*) \\
&\underset{\text{(g definition)}}{=} \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x) \right) \\
&\underset{\text{(inf definition)}}{\leq} f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} \nu_i^* h_i(x^*) \\
&\underset{(4)}{\leq} f_0(x^*),
\end{aligned}
$$

(4): (x^*, λ^*, ν^*) satisfies $\lambda^* \succeq 0$, $f_i(x^*) \leq 0$, and $h_i(x^*) = 0$.

...is complementary slackness

From previous slide,

$$\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = 0, \tag{5}$$

which is the condition of **complementary slackness**. This means

$$\begin{aligned} \lambda_i^* > 0 \quad &\Longrightarrow \quad f_i(x^*) = 0, \\ f_i(x^*) < 0 \quad &\Longrightarrow \quad \lambda_i^* = 0. \end{aligned}$$

From λ_i, read off which inequality constraints are strict.

KKT conditions for global optimum

Assume functions f_i, h_i are differentiable and **strong duality**. Since x^* minimizes $L(x, \lambda^*, \nu^*)$, derivative at x^* is zero,

$$\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0.$$

KKT conditions definition: we are at **global optimum**, $(x, \lambda, \nu) = (x^*, \lambda^*, \nu^*)$ when (a) **strong duality** holds, and (b)

$$\begin{aligned} f_i(x) &\leq 0, \ i = 1, \ldots, m \\ h_i(x) &= 0, \ i = 1, \ldots, p \\ \lambda_i &\geq 0, \ i = 1, \ldots, m \\ \lambda_i f_i(x) &= 0, \ i = 1, \ldots, m \\ \nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) &= 0 \end{aligned}$$

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KKT conditions for global optimum

In summary: if

- primal problem convex and
- inequality constraints affine

then strong duality holds. If in addition

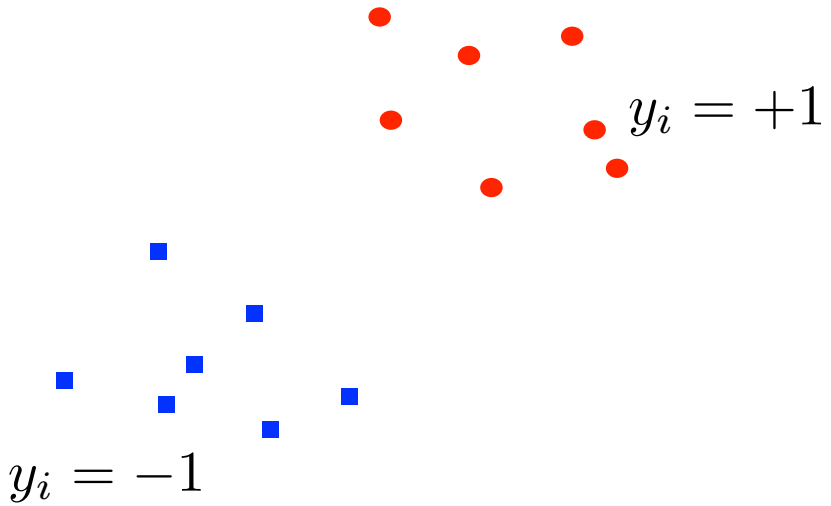
- functions f_i, h_i differentiable

then KKT conditions are *necessary and sufficient* for optimality.

Support vector classification

Linearly separable points

Classify two clouds of points, where there exists a hyperplane which linearly separates one cloud from the other without error.



Data given by $\{x_i, y_i\}_{i=1}^n$, $x_i \in \mathbb{R}^d$, $y_i \in \{-1, +1\}$

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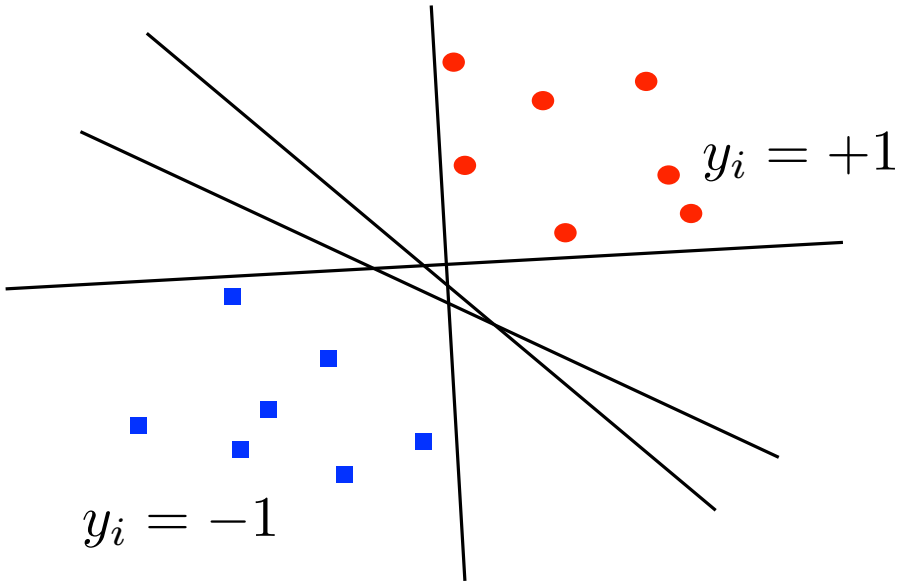

Hyperplane equation $w^\top x + b = 0$. Linear discriminant given by

$$w^\top x + b \begin{cases} \geq 0, & \text{class } +1 \\ < 0, & \text{class } -1. \end{cases}$$

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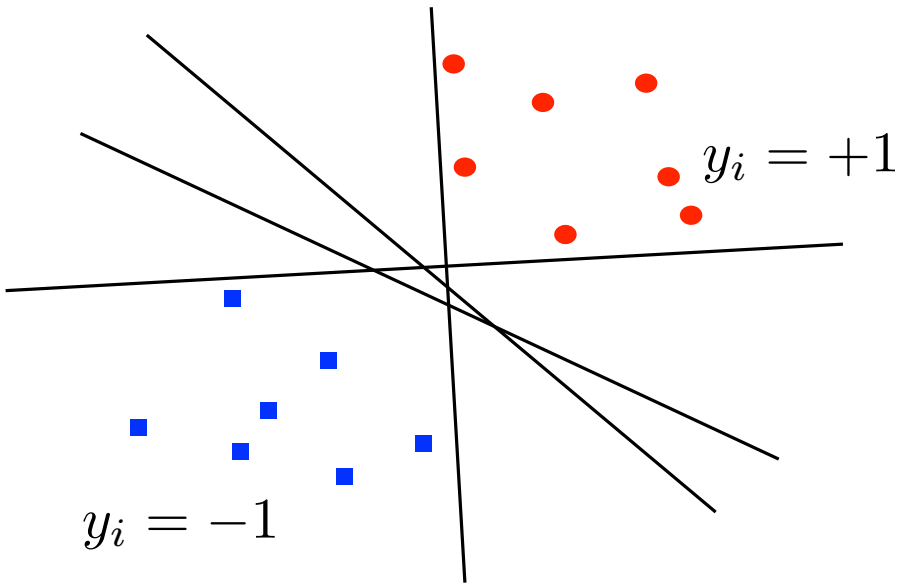

For a datapoint close to the decision boundary, a small change leads to a change in classification. Can we make the classifier more robust?

Linearly separable points

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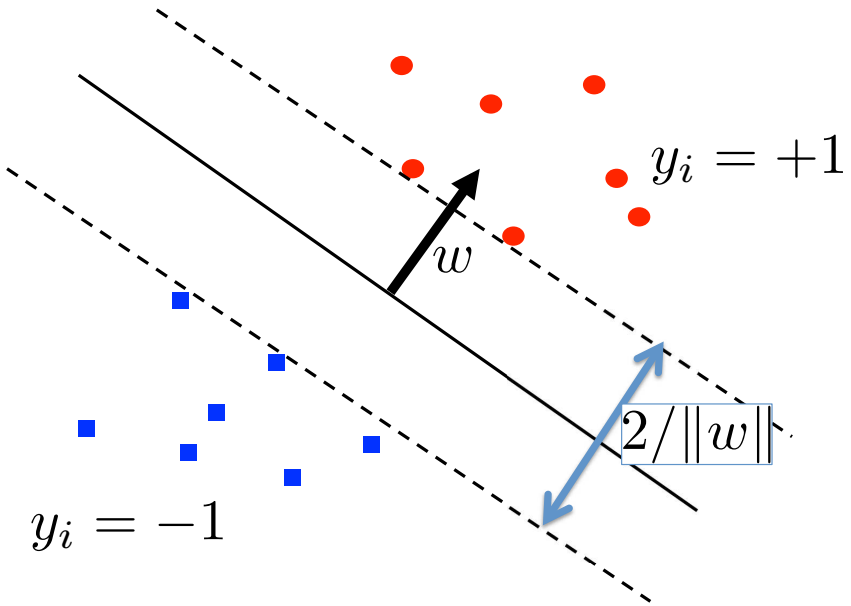

Smallest distance from each class to the separating hyperplane $w^\top x + b$ is called the **margin**.

Maximum margin classifier, linearly separable case

This problem can be expressed as follows:

$$\max_{w,b} (\text{margin}) = \max_{w,b} \left(\frac{1}{\|w\|} \right) \tag{6}$$

subject to

$$\begin{cases} w^\top x_i + b \geq 1 & i \ : \ y_i = +1, \\ w^\top x_i + b \leq -1 & i \ : \ y_i = -1. \end{cases} \tag{7}$$

The resulting classifier is

$$y = \text{sign}(w^\top x + b),$$

We can rewrite to obtain a *quadratic program*:

$$\min_{w,b} \frac{1}{2} \|w\|^2$$

subject to

$$y_i(w^\top x_i + b) \geq 1. \tag{8}$$

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Maximum margin classifier: with errors allowed

Allow "errors": points within the margin, or even on the wrong side of the decision boudary. Ideally:

$$\min_{w,b} \left(\frac{1}{2}\|w\|^2 + C \sum_{i=1}^{n} \mathbb{I}[y_i \left(w^\top x_i + b \right) < 0] \right),$$

where C controls the tradeoff between maximum margin and loss. Replace with **convex upper bound**:

$$\min_{w,b} \left(\frac{1}{2}\|w\|^2 + C \sum_{i=1}^{n} \theta \left(y_i \left(w^\top x_i + b \right) \right) \right).$$

with hinge loss,

$$\theta(\alpha) = (1-\alpha)_+ = \begin{cases} 1-\alpha, & 1-\alpha > 0 \\ 0, & \text{otherwise.} \end{cases}$$

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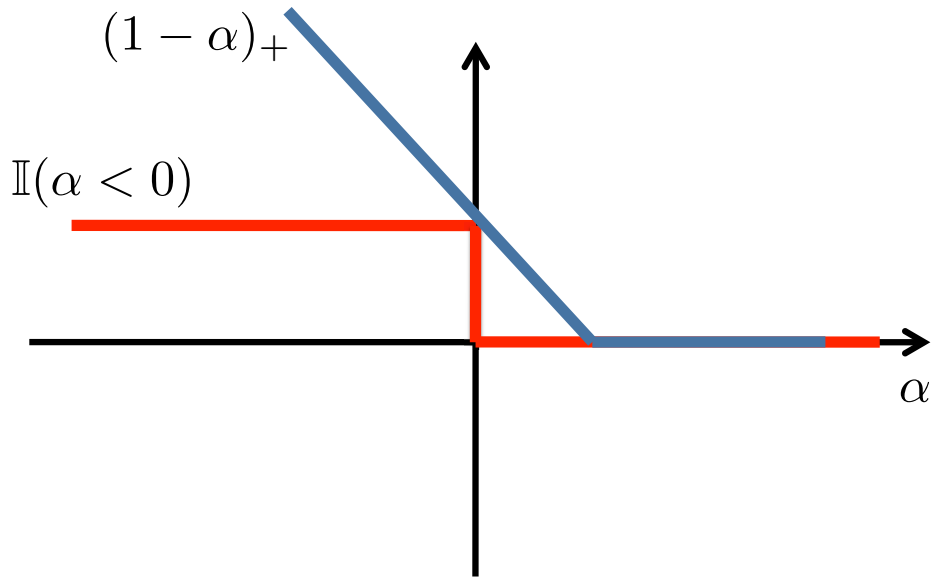

Support vector classification

Substituting in the hinge loss, we get

$$\min_{w,b} \left(\frac{1}{2}\|w\|^2 + C \sum_{i=1}^{n} \theta\left(y_i\left(w^\top x_i + b\right)\right)\right).$$

To simplify, use substitution $\xi_i = \theta\left(y_i\left(w^\top x_i + b\right)\right)$:

$$\min_{w,b,\xi} \left(\frac{1}{2}\|w\|^2 + C \sum_{i=1}^{n} \xi_i \right) \tag{9}$$

subject to

$$\xi_i \geq 0 \qquad y_i\left(w^\top x_i + b\right) \geq 1 - \xi_i$$

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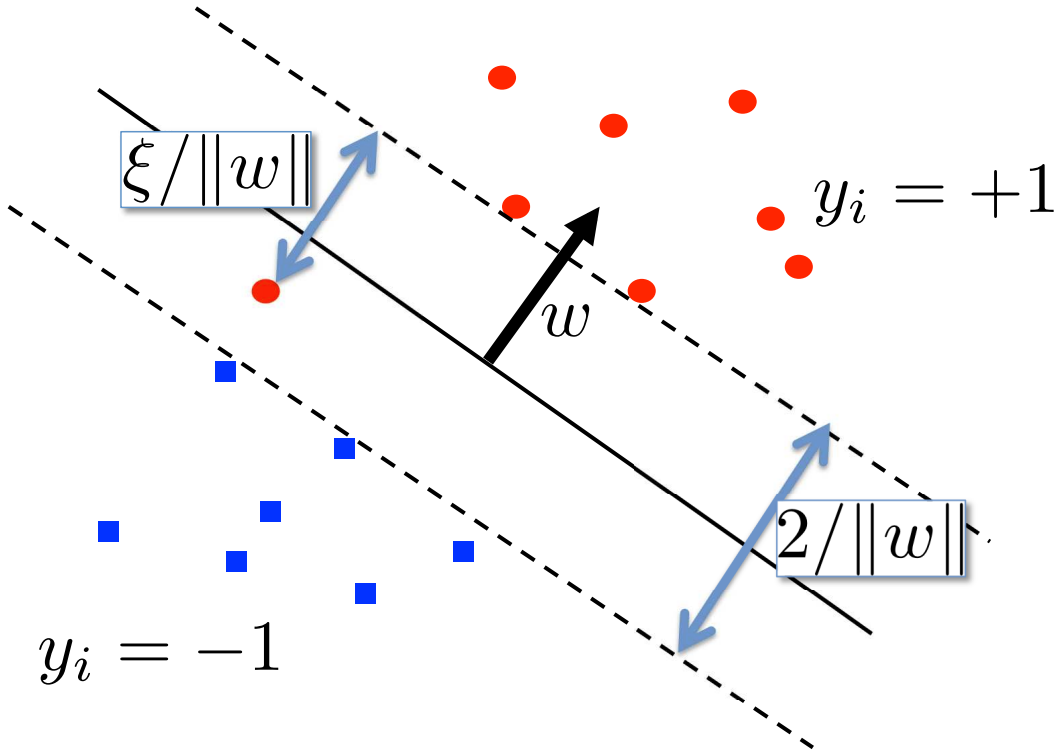

Does strong duality hold?

1. Is the optimization problem convex wrt the variables w, b, ξ?

$$\begin{aligned} \text{minimize} \quad & f_0(w, b, \xi) := \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n} \xi_i \\ \text{subject to} \quad & f_i(w, b, \xi) := 1 - \xi_i - y_i\left(w^\top x_i + b\right) \leq 0,\ i = 1, \ldots, n \\ & f_i(w, b, \xi) := -\xi_i \leq 0,\ i = n+1, \ldots, 2n \end{aligned}$$

   Each of $f_0, f_1, \ldots, f_n$ are convex. No equality constraints.

2. Does Slater's condition hold? Yes (trivially) since inequality constraints **affine**.

Thus **strong duality** holds, the problem is differentiable, hence the KKT conditions hold at the global optimum.

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Support vector classification: Lagrangian

The Lagrangian: $L(w, b, \xi, \alpha, \lambda) =$

$$\frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\xi_i + \sum_{i=1}^{n}\alpha_i\left(1 - \xi_i - y_i\left(w^\top x_i + b\right)\right) + \sum_{i=1}^{n}\lambda_i(-\xi_i)$$

with dual variable constraints

$$\alpha_i \geq 0, \qquad \lambda_i \geq 0.$$

Minimize wrt the primal variables w, b, and ξ.

Derivative wrt w:

$$\frac{\partial L}{\partial w} = w - \sum_{i=1}^{n}\alpha_i y_i x_i = 0 \qquad w = \sum_{i=1}^{n}\alpha_i y_i x_i. \tag{10}$$

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Since $\lambda_i \geq 0$,

$$\alpha_i \leq C.$$

Now use complementary slackness:

Non-margin SVs (margin errors): $\alpha_i = C > 0$:

1. We immediately have $y_i \left(w^\top x_i + b\right) = 1 - \xi_i$.
2. Also, from condition $\alpha_i = C - \lambda_i$, we have $\lambda_i = 0$, so $\xi_i \geq 0$

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The support vectors

We observe:

1. The solution is sparse: points which are neither on the margin nor "margin errors" have $\alpha_i = 0$
2. The support vectors: only those points on the decision boundary, or which are margin errors, contribute.
3. Influence of the non-margin SVs is bounded, since their weight cannot exceed C.

Support vector classification: dual function

Thus, our goal is to maximize the dual,

$$
\begin{aligned}
g(\alpha,\lambda) &= \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\xi_i + \sum_{i=1}^{n}\alpha_i\left(1 - y_i\left(w^\top x_i + b\right) - \xi_i\right) \\
&\quad + \sum_{i=1}^{n}\lambda_i(-\xi_i) \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j x_i^\top x_j + C\sum_{i=1}^{n}\xi_i - \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j x_i^\top x_j \\
&\quad - b\underbrace{\sum_{i=1}^{n}\alpha_i y_i}_{0} + \sum_{i=1}^{n}\alpha_i - \sum_{i=1}^{n}\alpha_i\xi_i - \sum_{i=1}^{n}(C-\alpha_i)\xi_i \\
&= \sum_{i=1}^{n}\alpha_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j x_i^\top x_j.
\end{aligned}
$$

Support vector classification: dual problem

Maximize the dual,

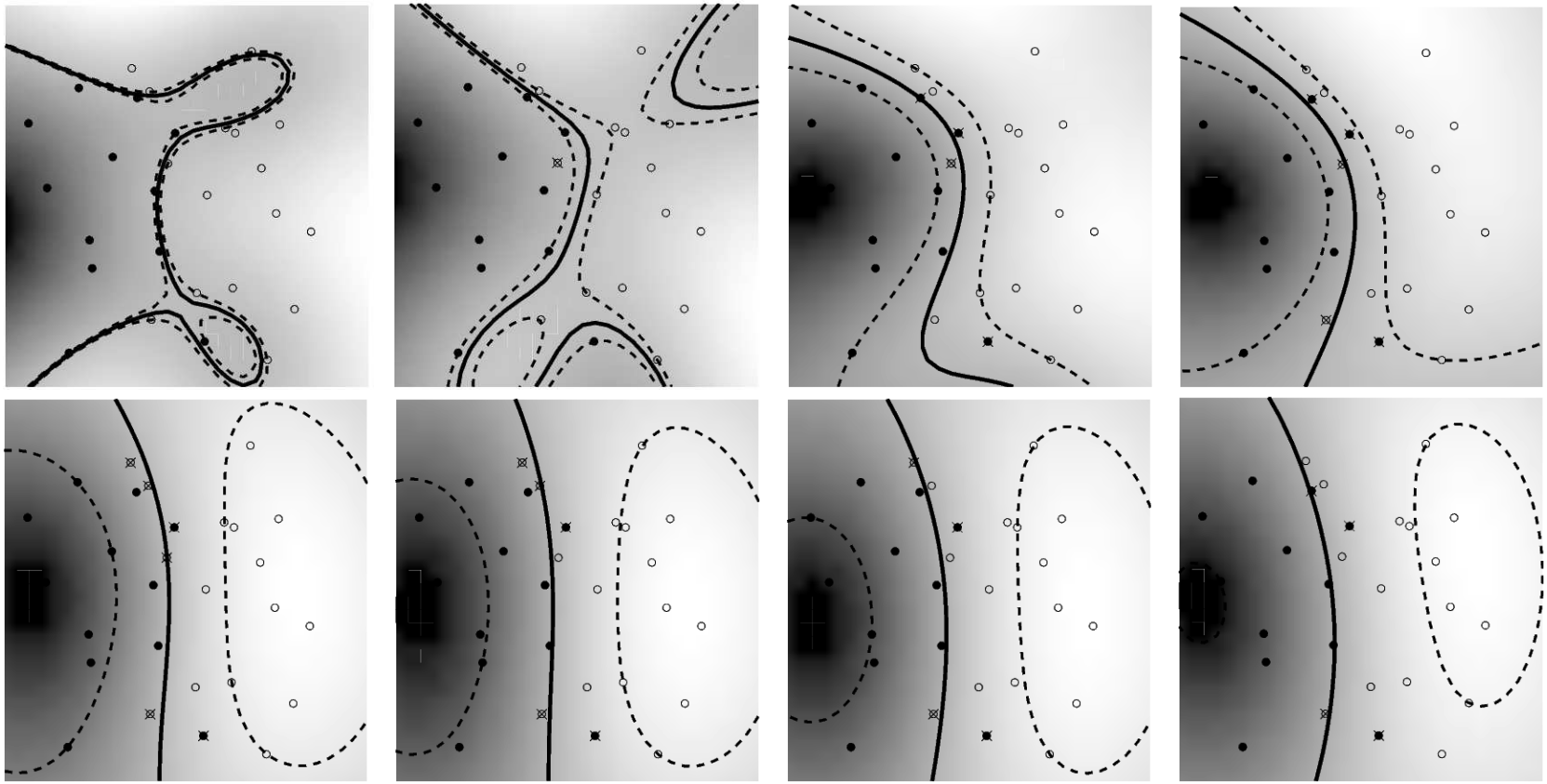
$$g(\alpha) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j x_i^\top x_j,$$

subject to the constraints

$$0 \leq \alpha_i \leq C, \quad \sum_{i=1}^{n} y_i \alpha_i = 0$$

This is a quadratic program. From α, obtain the hyperplane with $w = \sum_{i=1}^{n} \alpha_i y_i x_i$
Offset b can be obtained from any of the margin SVs:
$1 = y_i \left(w^\top x_i + b \right)$.

Support vector classification: kernel version



Taken from Schoelkopf and Smola (2002)

Support vector classification: kernel version

Maximum margin classifier in RKHS: write the hinge loss formulation

$$\min_{w} \left(\frac{1}{2} \|w\|_{\mathcal{H}}^2 + C \sum_{i=1}^{n} \theta \left(y_i \left\langle w, k(x_i, \cdot) \right\rangle_{\mathcal{H}} \right) \right)$$

for the RKHS $\mathcal{H}$ with kernel $k(x, x')$. Maximizing the margin equivalent to minimizing $\|w\|_{\mathcal{H}}^2$: for many RKHSs a smoothness constraint (e.g. Gaussian kernel).

Optimization over an infinitely dimensional space!

Support vector classification: kernel version

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Optimization over an infinitely dimensional space!

Support vector classification: kernel version

Dual in the linear case:

$$g(\alpha) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j x_i^\top x_j,$$

subject to the constraints

$$0 \leq \alpha_i \leq C, \quad \sum_{i=1}^{n} y_i \alpha_i = 0$$

Dual in the kernel case:

$$\max_{\alpha} \left(\sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j k(x_i, x_j) \right),$$

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Primal and the representer theorem

After solving the dual we can obtain the decision function

$$w(\cdot) = \sum_{i=1}^{n} y_i \alpha_i k(x_i, \cdot).$$

which lies in a finite dimensional subspace of $\mathcal{H}$, i.e., it is a (sparse) linear combination of the features (representer theorem). Thus, we can also derive the finite-dimensional primal by setting $w(\cdot) = \sum_{i=1}^{n} \beta_i k(x_i, \cdot)$.

$$\min_{\beta,\xi} \left(\frac{1}{2} \beta^\top K \beta + C \sum_{i=1}^{n} \xi_i \right) \tag{13}$$

where the matrix K has i,jth entry $K_{ij} = k(x_i, x_j)$, subject to

$$\xi_i \geq 0 \qquad y_i \sum_{j=1}^{n} \beta_j k(x_i, x_j) \geq 1 - \xi_i.$$

What is an advantage of the dual?

Questions?


THE KERNEL
BREWERY
LONDON
BALTIC PORTER
7.3% ABV
TH
THE KERNEL
EXPORT STOUT
LONDON 1889
7.4% ABV

Representer theorem

Learning problem: setting

Given a set of paired observations $(x_1, y_1), \ldots (x_n, y_n)$ (regression or classification).
Find the function f^* in the RKHS $\mathcal{H}$ which satisfies

$$J(f^*) = \min_{f \in \mathcal{H}} J(f), \tag{14}$$

where

$$J(f) = L_y(f(x_1), \ldots, f(x_n)) + \Omega\left(\|f\|^2_{\mathcal{H}}\right),$$

Ω is non-decreasing, and y is the vector of y_i.

- Classification: $L_y(f(x_1), \ldots, f(x_n)) = \sum_{i=1}^n \mathbb{I}_{y_i f(x_i) \leq 0}$
- Regression: $L_y(f(x_1), \ldots, f(x_n)) = \sum_{i=1}^n (y_i - f(x_i))^2$

Representer theorem

The representer theorem: solution to

$$\min_{f \in \mathcal{H}} \left[L_y(f(x_1), \ldots, f(x_n)) + \Omega\left(\|f\|_{\mathcal{H}}^2 \right) \right]$$

takes the form

$$f^* = \sum_{i=1}^{n} \alpha_i k(x_i, \cdot).$$

If Ω is strictly increasing, all solutions have this form.

Representer theorem: proof

Proof: Denote f_s projection of f onto the subspace

$$\operatorname{span}\left\{k(x_i,\cdot):\ 1\leq i\leq n\right\}, \tag{15}$$

such that

$$f = f_s + f_\perp,$$

where $f_s = \sum_{i=1}^{n} \alpha_i k(x_i,\cdot)$.
Regularizer:

$$\|f\|_{\mathcal{H}}^2 = \|f_s\|_{\mathcal{H}}^2 + \|f_\perp\|_{\mathcal{H}}^2 \geq \|f_s\|_{\mathcal{H}}^2,$$

then

$$\Omega\left(\|f\|_{\mathcal{H}}^2\right) \geq \Omega\left(\|f_s\|_{\mathcal{H}}^2\right),$$

so this term is minimized for $f = f_s$.

Representer theorem: proof

Proof (cont.): Individual terms $f(x_i)$ in the loss:

$$f(x_i) = \langle f, k(x_i, \cdot) \rangle_{\mathcal{H}} = \langle f_s + f_\perp, k(x_i, \cdot) \rangle_{\mathcal{H}} = \langle f_s, k(x_i, \cdot) \rangle_{\mathcal{H}},$$

so

$$L_y(f(x_1), \ldots, f(x_n)) = L_y(f_s(x_1), \ldots, f_s(x_n)).$$

Hence

- Loss $L(\ldots)$ only depends on the component of f in the data subspace,
- Regularizer $\Omega(\ldots)$ minimized when $f = f_s$.
- If Ω is strictly non-decreasing, then $\|f_\perp\|_{\mathcal{H}} = 0$ is required at the minimum.

ν-SVM

Support vector classification: the ν-SVM

Parameter C in SVMs can be hard to interpret. Modify the formulation to get a more intuitive parameter ν:

$$\min_{w,\rho,\xi} \left(\frac{1}{2}\|w\|^2 - \nu\rho + \frac{1}{n}\sum_{i=1}^{n} \xi_i \right)$$

subject to

$$\begin{aligned} \rho &\geq 0 \\ \xi_i &\geq 0 \\ y_i w^\top x_i &\geq \rho - \xi_i, \end{aligned}$$

(now we directly adjust the constraint threshold ρ).

The ν-SVM: Lagrangian

$$\frac{1}{2}\|w\|_{\mathcal{H}}^2+\frac{1}{n}\sum_{i=1}^{n}\xi_i-\nu\rho+\sum_{i=1}^{n}\alpha_i\left(\rho-y_i w^\top x_i-\xi_i\right)+\sum_{i=1}^{n}\beta_i(-\xi_i)+\gamma(-\rho)$$

for dual variables $\alpha_i \geq 0$, $\beta_i \geq 0$, and $\gamma \geq 0$.

Differentiating and setting to zero for each of the primal variables w, ξ, ρ,

$$w = \sum_{i=1}^{n}\alpha_i y_i x_i$$

$$\alpha_i+\beta_i = \frac{1}{n} \tag{16}$$

$$\nu = \sum_{i=1}^{n}\alpha_i-\gamma \tag{17}$$

From $\beta_i \geq 0$, equation (16) implies

$$0 \leq \alpha_i \leq 1/n.$$

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for dual variables $\alpha_i \geq 0$, $\beta_i \geq 0$, and $\gamma \geq 0$.
Differentiating and setting to zero for each of the primal variables w, ξ, ρ,

$$\begin{aligned} w &= \sum_{i=1}^{n}\alpha_i y_i x_i \\ \alpha_i+\beta_i &= \frac{1}{n} \qquad (16) \\ \nu &= \sum_{i=1}^{n}\alpha_i-\gamma \qquad (17) \end{aligned}$$

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Complementary slackness (1)

Complementary slackness conditions:

Assume $\rho > 0$ at the global solution, hence $\gamma = 0$, and

$$\sum_{i=1}^{n} \alpha_i = \nu. \tag{18}$$

Case of $\xi_i > 0$: complementary slackness states $\beta_i = 0$, hence from (16) we have $\alpha_i = 1/n$. Denote this set as $N(\alpha)$. Then

$$\sum_{i \in N(\alpha)} \frac{1}{n} = \sum_{i \in N(\alpha)} \alpha_i \leq \sum_{i=1}^{n} \alpha_i = \nu,$$

so

$$\frac{|N(\alpha)|}{n} \leq \nu,$$

and ν is an upper bound on the proportion of non-margin SVs.

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Complementary slackness (2)

Case of $\xi_i = 0$: $\alpha_i < 1/n$. Denote by $M(\alpha)$ the set of points $1/n > \alpha_i > 0$. Then from (18),

$$\nu = \sum_{i=1}^{n} \alpha_i = \sum_{i \in N(\alpha)} \frac{1}{n} + \sum_{i \in M(\alpha)} \alpha_i \leq \sum_{i \in M(\alpha) \cup N(\alpha)} \frac{1}{n},$$

thus

$$\nu \leq \frac{|N(\alpha)| + |M(\alpha)|}{n},$$

and ν is a lower bound on the proportion of support vectors with non-zero weight (both on the margin, and "margin errors").

Dual for ν-SVM

Substituting into the Lagrangian, we get

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^\top x_j + \frac{1}{n}\sum_{i=1}^{n}\xi_i - \rho\nu - \sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^\top x_j$$

$$+\sum_{i=1}^{n}\alpha_i\rho - \sum_{i=1}^{n}\alpha_i\xi_i - \sum_{i=1}^{n}\left(\frac{1}{n}-\alpha_i\right)\xi_i - \rho\left(\sum_{i=1}^{n}\alpha_i - \nu\right)$$

$$= -\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^\top x_j$$

Maximize:

$$g(\alpha) = -\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^\top x_j,$$

subject to

$$\sum_{i=1}^{n}\alpha_i \geq \nu \qquad 0 \leq \alpha_i \leq \frac{1}{n}.$$

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$$g(\alpha) = -\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^\top x_j,$$

subject to

$$\sum_{i=1}^{n}\alpha_i \geq \nu \qquad 0 \leq \alpha_i \leq \frac{1}{n}.$$