Combinatorial Channel Signature Modulation for Wireless Networks

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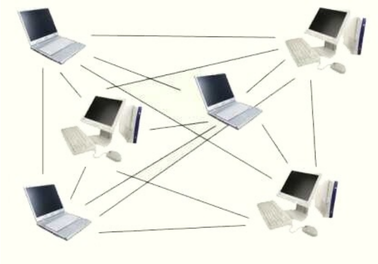
Problem outline

- wireless mutual broadcast with $N+1$ half-duplex nodes
- each node has a k-bit message to transmit to all others
- Can all nodes transmit at the same frequency without elaborate time scheduling?



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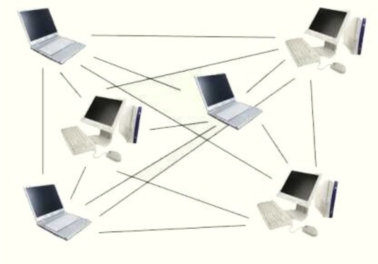

Introduction

CCSM (Combinatorial Channel Signature Modulation):

- sparse, combinatorial representation of messages
- compressed sensing based decoding
- minimal MAC Layer coordination: access to a shared channel
  - no need for collision detection/avoidance scheme (CSMA/CA)
- robust to time dispersion
  - no need for guard intervals to eliminate ISI

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Overview

CS for multiterminal communications

Combinatorial encoder

Sparse recovery solver

Simulation results

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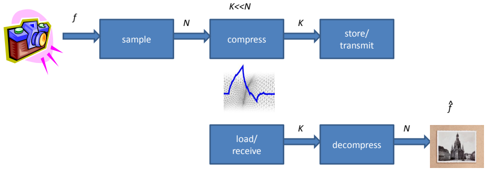
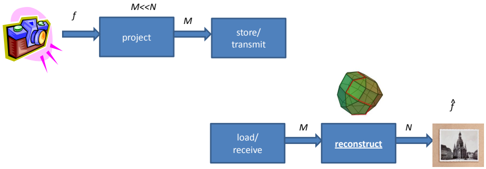
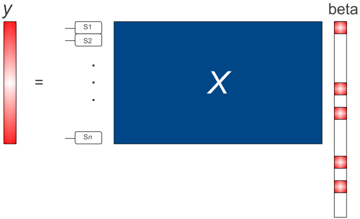
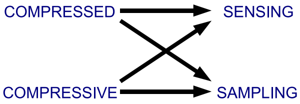
Compressed sensing

- Compressed sensing (Candes, Romberg and Tao 2006; Donoho 2006) combines sampling and compression steps into one - into taking random linear projections.









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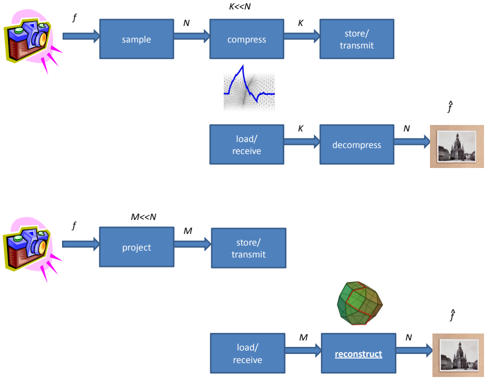

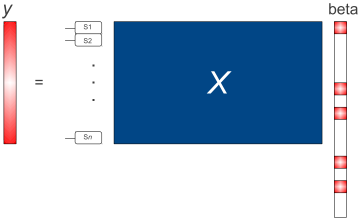

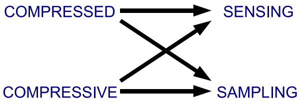

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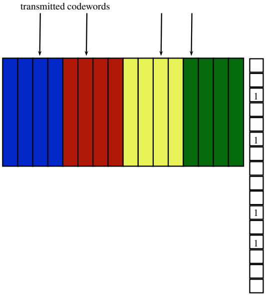
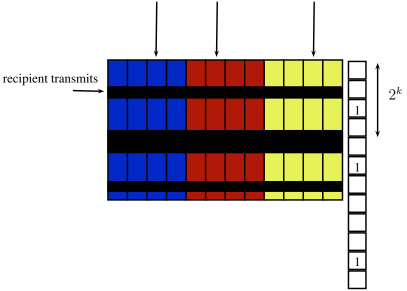
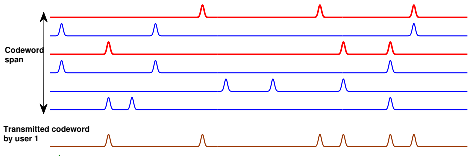
Embracing the additive nature of wireless

- (Zhang and Guo 2010): each node is assigned a (known) dictionary of (sparse) on-off signalling codewords, each codeword corresponding to a single message
- Own transmissions seen as erasures in dictionary
- The dictionary size exponential in the number of bits k in a message - sparse recovery problem of size $2^k N$





Combinations of the codebook elements?

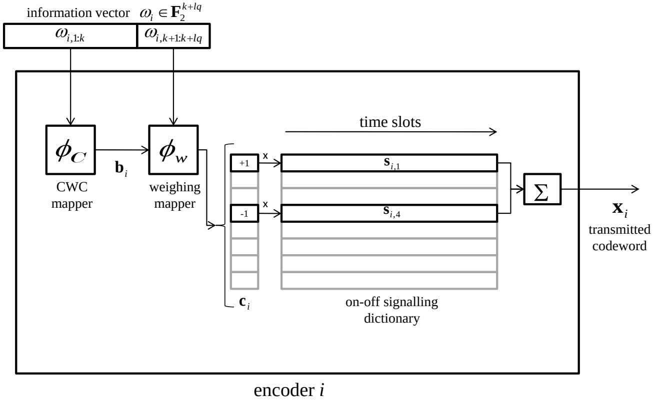


- Idea: Transmit (weighted) sums of a fixed number l of on-off signalling codewords.
- the *choice* of the l-combination of the dictionary elements carries information
- Requires efficient encoding of messages into l-combinations: **constant weight coding** (l out of L codes)
- Need $l \ll L$ for sparse recovery

Reduction of complexity

- (Zhang and Guo, 2010): $k = \log_2 L$ - sparse recovery problem of size $2^k N$
- CCSM: $k \approx \log_2 \binom{L}{l} = \mathcal{O}(L^\alpha \log_2 L)$, for $l = L^\alpha$, $0 < \alpha < 1$ - sparse recovery problem of size$\sim k^{1/\alpha} N$

Encoder



$$\phi_{\mathcal{C}} \; : \; 00000 \;\mapsto\; 010100000$$

$$\phi_{\mathcal{C}} \; : \; 00001 \;\mapsto\; 000101000$$

$$\cdots$$

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Constant weight coding

- Combinatorial representation of information
- using l-combinations of the set of L elements, i.e., length-L binary vectors with exactly l ones.
    - $l \ll L$ (sparse)

Definition

An (L, l)-constant weight code is a set of length-L binary vectors with Hamming weight l:

$$\mathcal{C} \subseteq \{c \in \mathbb{F}_2^L : \ w_H(c) = l\}.$$

- single-bit error/single-type error detection
- two-out-of-five barcode

0 1 2 3 4 5 6 7 8 9

Constant weight coding

- Set of possible messages: $\mathcal{S} = \{0, 1 \ldots, K-1\}$. Usually $K = 2^k$, and we identify $\mathcal{S} \leftrightarrow \mathbb{F}_2^k$.
- An (L, l)-constant weight code is $\mathcal{C} \subseteq \{c \in \mathbb{F}_2^L : w_H(c) = l\}$.
- Goal: construct a bijective map $\phi_{\mathcal{C}} : \mathcal{S} \to \mathcal{C}$

- Enumeration approaches:
  - (Schalkwijk 1972), (Cover 1973): lexicographic ordering of codewords, requires registers of length $\mathcal{O}(L)$.
  - (Ramabadran 1990): arithmetic coding, computational complexity $\mathcal{O}(L)$.
  - (Knuth 1986): complementation method, computational complexity $\mathcal{O}(L)$ - but much faster in practice. Works only for balanced codes: $l = \lfloor L/2 \rfloor$.

- Can we do better when $l \ll L$?
  - (Tian, Vaishampayan, Sloane 2009): embed both $\mathcal{S}$ and $\mathcal{C}$ into $\mathbb{R}^l$ and establish bijective maps by dissecting certain polytopes in $\mathbb{R}^l$.

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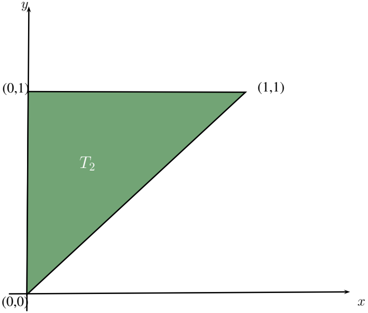
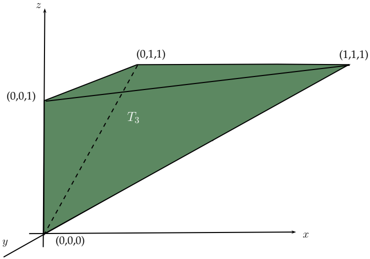
Embedding $\mathcal{C}$ into $\mathbb{R}^l$

- Given a constant-weight codeword c, define $\phi(c) = (\frac{y_1}{L}, \ldots, \frac{y_l}{L}) \in \mathbb{R}^l$, with y_i:= position of the i-th 1 in c.
- for $L = 5$, $l = 2$, $\phi(01010) = (2/5, 4/5)$.
- $\phi(\mathcal{C})$ is a discrete subset of the convex hull T_l of the points:

$$\begin{array}{ccccc}
(0, & 0, & \ldots & 0, & 0) \\
(0, & 0, & \ldots & 0, & 1) \\
\vdots & \vdots & \ldots & \vdots & \vdots \\
(0, & 1, & \ldots & 1, & 1) \\
(1, & 1, & \ldots & 1, & 1)
\end{array}$$

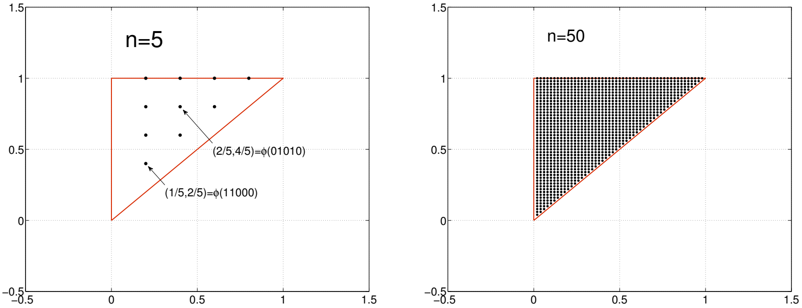
$\mathcal{C}$ embedded in a tetrahedron





- T_2 is the right triangle with vertices $(0,0)$, $(0,1)$, $(1,1)$
- $Vol(T_l) = \frac{1}{l!}$ (the unit cube can be split into $l!$ tetrahedra congruent to T_l)

The case $l = 2$

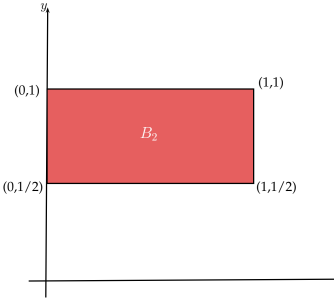


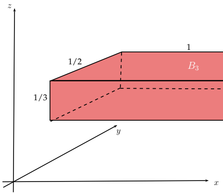
Assume: information is a brick

- Brick $B_l \subset \mathbb{R}^l$ is a hyper-rectangle

$$B_l := [0,1]\times[\frac{1}{2},1]\times[\frac{2}{3},1]\times\cdots\times[\frac{l-1}{l},1]$$

- Represent messages $s \in \{0,1,\ldots,K-1\}$, $K \leq \binom{L}{l}$ as integer l-tuple $(b_1^{(s)}, b_2^{(s)}, \ldots, b_l^{(s)})$, s.t. $(\frac{b_1^{(s)}}{n}, \frac{b_2^{(s)}}{n}, \ldots, \frac{b_l^{(s)}}{n}) \in B_l$: quotient and remainder
- $Vol(B_l) = \frac{1}{l!} = Vol(T_l)$



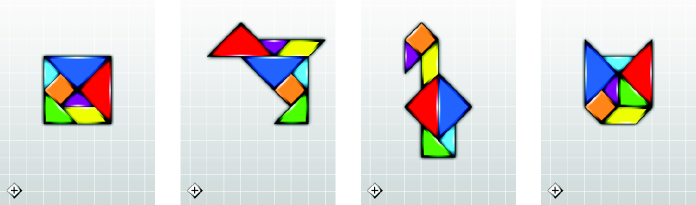


Dissections

Hilbert's third problem:

For any two polyhedra of the same volume, is it possible to dissect one into a finite number of pieces that can be rearranged to give the other?

- In $l = 2$ dimensions (Bolyai-Gerwien 1833): yes



"scissor-equivalence"

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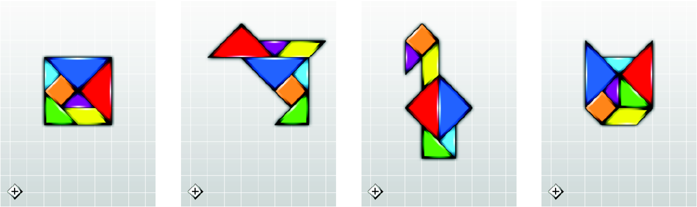

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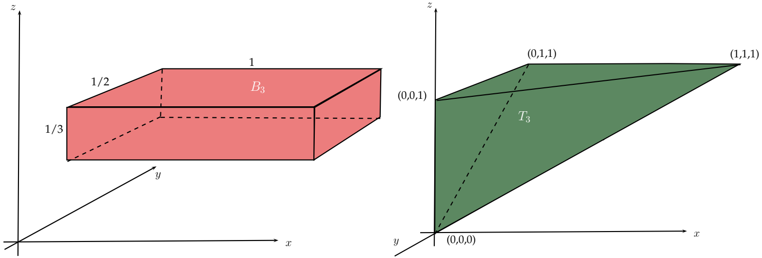

- messages $\mathcal{S} = \{0, 1 \ldots, K-1\} \longleftrightarrow B_l \subset \mathbb{R}^l \underset{\longleftrightarrow}{diss.} T_l \subset \mathbb{R}^l$ $\longleftrightarrow$ constant weight code $\mathcal{C}$
- (Tian, Vaishampayan, Sloane 2009) give an explicit recursive dissection of B_l into T_l computable in $\mathcal{O}(l^2)$

Rate

- CWC rate: $R(\mathcal{C}) = \frac{1}{L}\log_2|\mathcal{C}| \le \frac{1}{L}\log_2\binom{L}{l} = \mathcal{O}(l\frac{\log_2 L}{L}) \to 0$, $L \to \infty$, when $l \ll L$.

$CCSM\ rate \neq CWC\ rate$

- CCSM rate: $\frac{N}{M}\left(\log_2\binom{L}{l} + lq\right)$, where
  - M is the time duration of the waveforms (number of rows in the CS problem + number of own transmissions)
- Typical CS results: it suffices to take M to be the number of non-zeros $\times$ log (size of the vector)
- $M = \mathcal{O}(lN\log_2(LN)) \Rightarrow CCSM\ rate = \mathcal{O}(\frac{\log_2 L}{\log_2 L + \log_2 N})$

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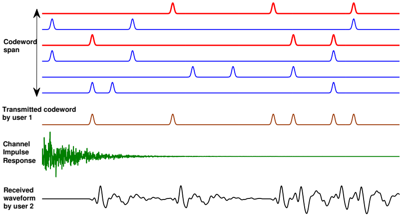
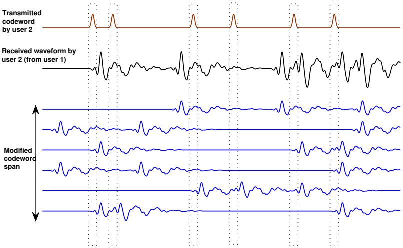
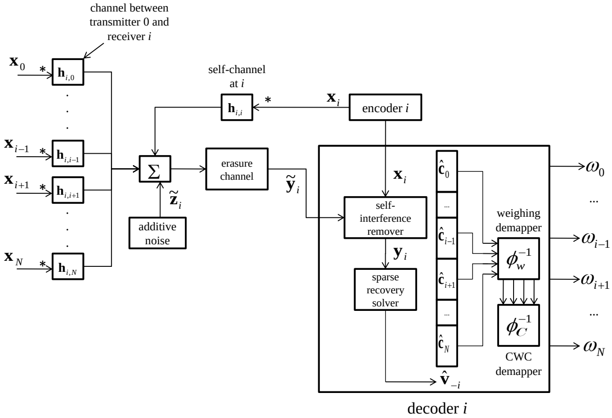
Channel signatures





- Each user convolves codebook elements with the channel impulse response
- Can subtract echoes of its own transmissions (self-interference remover)

Decoder



$$\mathbf{x}_i = \mathbf{S}_i \mathbf{c}_i$$

$$\mathbf{y}_i = \mathbf{E}_i \left(\sum_{j=0}^{N} \mathbf{h}_{i,j} * \mathbf{S}_j \mathbf{c}_j + \tilde{\mathbf{z}}_i \right) - \mathbf{E}_i \left(\mathbf{h}_{i,i} * \mathbf{S}_i \mathbf{c}_i \right) = \mathbf{A}_{-i} \mathbf{v}_{-i} + \mathbf{z}_i$$

Sparse recovery solver

$$\mathbf{y}_i = \mathbf{E}_i \left(\sum_{j=0}^{N} \mathbf{h}_{i,j} * \mathbf{S}_j \mathbf{c}_j + \tilde{\mathbf{z}}_i \right) - \mathbf{E}_i \left(\mathbf{h}_{i,i} * \mathbf{S}_i \mathbf{c}_i \right) = \mathbf{A}_{-i} \mathbf{v}_{-i} + \mathbf{z}_i$$

- User i needs to solve the following problem to detect the desired signal:

$$\begin{aligned} \hat{\mathbf{v}}_{-i} &= \arg\min_{\mathbf{v}_{-i}} \|\mathbf{y}_i - \mathbf{A}_{-i}\mathbf{v}_{-i}\|_2 \\ \text{s.t. } \|\mathbf{c}_j\|_0 &= l, \text{ for all } j \neq i, \end{aligned}$$

  - A non-convex optimisation problem.
  - Exactly Nl out of NL entries in $\mathbf{v}_{-i}$ are non-zero - but sparsity level is: $\frac{l}{L} \ll 1$
  - Can use any of the myriad of CS decoding algorithms.

Sparse recovery solver

$$\mathbf{y}_i = \mathbf{E}_i \left(\sum_{j=0}^{N} \mathbf{h}_{i,j} * \mathbf{S}_j \mathbf{c}_j + \tilde{\mathbf{z}}_i \right) - \mathbf{E}_i \left(\mathbf{h}_{i,i} * \mathbf{S}_i \mathbf{c}_i \right) = \mathbf{A}_{-i} \mathbf{v}_{-i} + \mathbf{z}_i$$

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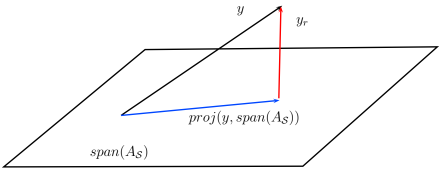
Sparse recovery solver

- Basis Pursuit/LASSO/convex relaxation (Tibshirani 1996), (Chen, Donoho, and Saunders 1998)
- LARS / homotopy (Efron et al, 2004)
- Greedy iterative methods:
  - Compressive Sampling Matching Pursuit - CoSaMP (Needell and Tropp 2009)
  - Subspace Pursuit - SP (Dai and Milenkovic 2009)

Subspace Pursuit

Greedily searches for the support set $\mathcal{S}$ such that y is closest to $span(A_{\mathcal{S}})$.

1. Initialise. Set $\mathcal{S}$ to the Nl columns that maximize $|\langle a_i, y\rangle|$
2. Identify further candidates. Set $\mathcal{S}'$ to the Nl columns that maximize $|\langle a_i, y - proj(y, span(A_{\mathcal{S}})\rangle|$
3. Merge and Prune. Set $\mathcal{S}$ to the Nl columns from $\mathcal{S} \cup \mathcal{S}'$ with largest magnitudes in $A^{+}_{\mathcal{S}\cup\mathcal{S}'}y$
4. Iterate (2)-(3) until the stopping criterion holds.



Group Subspace Pursuit

Sparse vector has additional structure: each of N subvectors has exactly l non-zero entries.

1. Initialise. Set $\mathcal{S}$ to the union of sets of l columns within each group of L that maximize $|\langle a_i, y\rangle|$
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Group Subspace Pursuit (2)

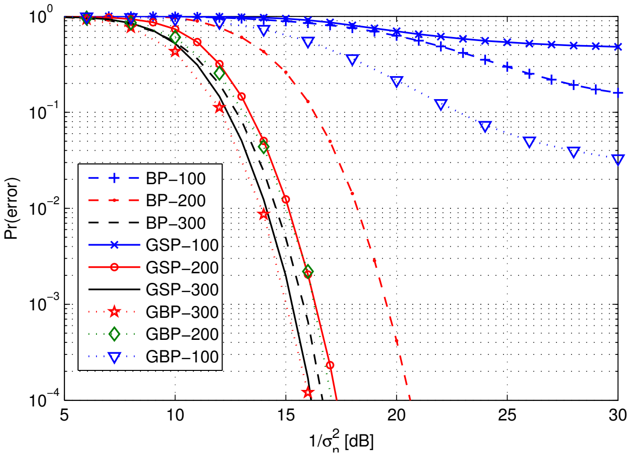


Figure: Performance comparison of Basis Pursuit/Lasso (BP), Group Basis Pursuit (GBP) and Group Subspace Pursuit (GSP). $N = 10$, $l = 4$, $L = 32$

CCSM Performance

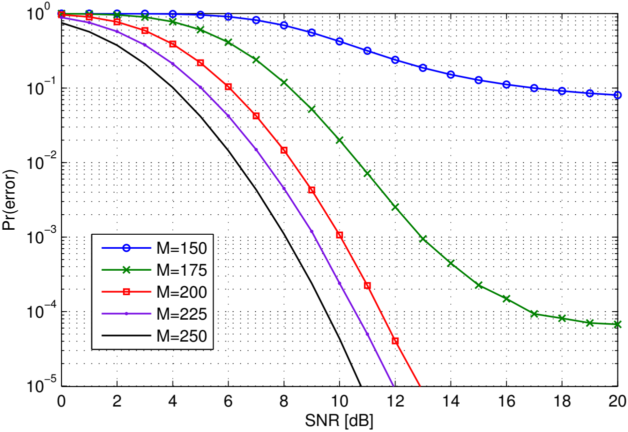
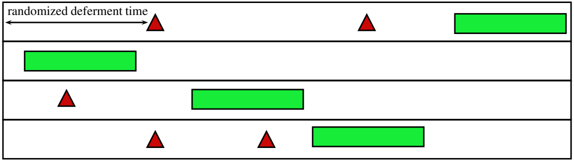


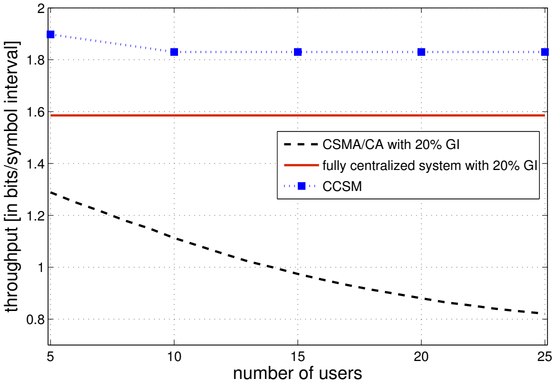
Figure: Performance of the proposed method in terms of message error rate: In this case there are $(N+1) = 5$ users simultaneously broadcasting messages using CCSM with $L = 64$, $l = 12$.

Comparison to CSMA/CA and TDMA (1)

1. TDMA
   - central controlling mechanism
   - time divided equally - no collisions, performance independent of the number of nodes
   - guard interval (cyclic prefix) of 20% slot duration
2. CSMA/CA
   - randomised deferment of transmissions in order to avoid collisions; contention window: 16-1024 symbol intervals
   - no symbol intervals wasted on distributed or short interframe space (DIFS/SIFS), propagation delay, physical or MAC message headers and ACK responses
   - a single message in each transmission queue
   - guard interval (cyclic prefix) of 20% slot duration



Comparison to CSMA/CA and TDMA (2)



CCSM: the minimum number of symbol intervals at which no message errors occurred in at least 100,000 simulation trials

$$L = 64,\ l = 12,\ q = 2 \text{ (QPSK)}$$

$$k = \left\lfloor \log_2 \binom{L}{l} \right\rfloor + lq = 65$$

Concluding remarks

- a novel decentralized modulation and multiplexing method for wireless networks
  - effective time/frequency duplex
  - minimal MAC
  - inherent robustness to time dispersion
- low computational complexity which takes advantage of
  - combinatorial representation of messages
  - sparse recovery detection
- significant throughput improvement in comparison to collision avoidance schemes

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References

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Signalling dictionary

- One construction of on-off signalling dictionary:
  - all columns of $\mathbf{S}_i$ have equal number of non-zero entries, set to $\lfloor \frac{M}{L} \rfloor$
  - every two columns in $\mathbf{S}_i$ have disjoint support
  - non-zero entries selected uniformly at random from some discrete constellation
- transmitted codeword $\mathbf{x}_i$ has exactly $l \cdot \lfloor \frac{M}{L} \rfloor$ non-zero entries (on-slots)
- $\tilde{M} = M - l \cdot \lfloor \frac{M}{L} \rfloor$ off-slots used to listen to the incoming signals
- Can also use LDPC / regular Gallager constructions (Baron, Sarvotham, Baraniuk 2010)

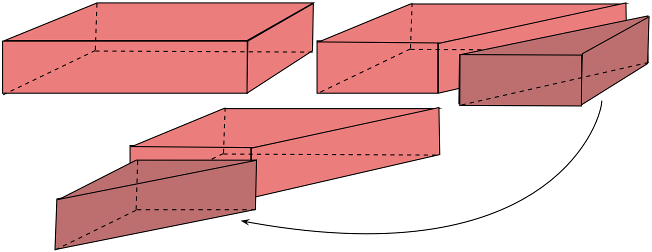
Assume: information is a brick

- Brick $B_l \subset \mathbb{R}^l$ is a hyper-rectangle
  $B_l := [0,1] \times [\frac{1}{2}, 1] \times [\frac{2}{3}, 1] \times \cdots \times [\frac{l-1}{l}, 1]$
- Let $l = 2$, and L even. Then one can uniquely represent message indices $s \in \{0, 1, \ldots, K-1\}$ (where $K \leq \frac{L(L-1)}{2}$) as integer pairs $(b_1^{(s)}, b_2^{(s)})$, where

$$\begin{aligned} 0 &\leq \alpha = & \left\lfloor \frac{s}{L/2} \right\rfloor \leq L-1 \\ 0 &\leq \beta = & s - \frac{L}{2}\alpha \leq \frac{L}{2} - 1 \end{aligned}$$

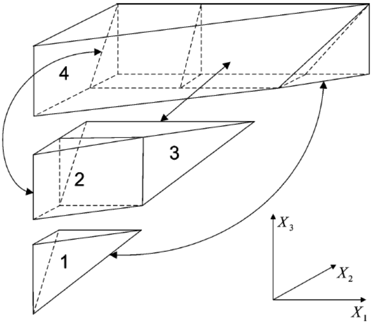
- Define $b_1^{(s)} = \alpha$, $b_2^{(s)} = \beta + \frac{L}{2}$.
- It follows that $\left\{ (\frac{b_1^{(s)}}{L}, \frac{b_2^{(s)}}{L}) \right\}_{s=0}^{K-1} \subset B_l$

Step 1: brick to triangular prism



$$B_3 = B_2 \times [2/3, 1] \to T_2 \times [2/3, 1]$$

Step 2: triangular prism to tetrahedron



- inductive dissection for general w:
  1. $B_w = B_{w-1} \times [\frac{w-1}{w}, 1] \to T_{w-1} \times [\frac{w-1}{w}, 1]$
  2. $T_{w-1} \times [\frac{w-1}{w}, 1] \to T_w$

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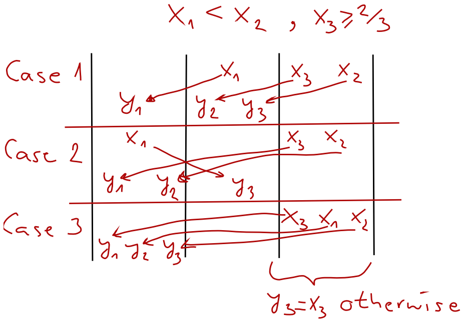

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- Some loss in rate due to rounding (when n is in the range 100-1000, and $w = \sqrt{n}$, the loss is 1-2 bits/block)